

No small nondeterministic read-once branching programs for CNFs of bounded treewidth

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Abstract. In this paper, given a parameter k , we demonstrate an infinite class of CNFs of treewidth at most k of their primary graphs such that the equivalent nondeterministic read-once branching programs (NROBPs) are of size at least n^{ck} for some universal constant c . Thus we rule out the possibility of fixed-parameter space complexity of NROBPs parameterized by the smallest treewidth of the equivalent CNF.

1 Introduction

Read-once Branching Programs (ROBPs) is a well known representation of Boolean functions. Oblivious ROBPs, better known as Ordered Binary Decision Diagrams (OBDDs), is a subclass of ROBPs, very well known because of its applications in the area of verification [2]. An important procedure in these applications is transformation of a CNF into an equivalent OBDD. The resulting OBDD can be exponentially larger than the initial CNF, however a space efficient transformation is possible for special classes of functions. For example, it has been shown in [3] that a CNF of treewidth k of its primal graph can be transformed into an OBDD of size $O(n^k)$. A natural question is if the upper bound can be made fixed-parameter i.e. of the form $f(k)n^c$ for some constant c . In [7] we showed that it is impossible by demonstrating that for each sufficiently large k there is an infinite class of CNFs of treewidth at most k whose smallest OBDD is of size at least $n^{k/5}$.

In this paper we report a follow up result showing that essentially the same lower bound holds for Non-deterministic ROBPs (NROBPs). In particular we show that there is a constant $0 < c < 1$ such that for each sufficiently large k there is an infinite class of CNFs of treewidth at most k (of their primary graphs) for which the space complexity of the equivalent NROBPs is at least n^{ck} . Note that NROBPs are strictly more powerful than ROBPs in the sense that there is an infinite class of functions having a poly-size NROBP representation and exponential ROBP space complexity [4]. In the same sense, ROBPs are strictly more powerful than OBDDs, hence the result proposed in this paper is a significant enhancement of the result of [7].

We believe this result is interesting from the parameterized complexity theory perspective because it contributes to the understanding of parameterized *space* complexity of various representations of Boolean functions. In particular, the proposed result implies that ROBPs are inherently incapable to efficiently represent functions that are representable by CNFs of bounded treewidth. A natural question for further research is

the space complexity of read c -times branching programs [1] (for an arbitrary constant c independent on k) w.r.t. the same class of functions.

To prove the proposed result, we use monotone 2-CNFs (their clauses are of form $(x_1 \vee x_2)$ where x_1 and x_2 are 2 distinct variables). These CNFs are in one-to-one correspondence with graphs having no isolated vertices: variables correspond to vertices and 2 variables occur in the same clause if and only if the corresponding vertices are adjacent. This correspondence allows us to use these CNFs and graphs interchangeably. We introduce the notion of Distant Matching Width (DMW) of a graph G and prove 2 theorems. One of them states that a NROBP equivalent to a monotone 2-CNF with the corresponding graph G having DMW at least t is of size at least $2^{t/a}$ where a is a constant dependent on the max-degree of G . The second theorem states that for each sufficiently large k there is an infinite family of graphs of treewidth k and max-degree 5 whose DMW is at least $b * \log n * k$ for some constant b independent of k . The main theorem immediately follows from replacement of t in the former lower bound by the latter one.

The strategy outlined above is similar to that we used in [7]. However, there are two essential differences. First, due to a much more ‘elusive’ nature of NORBPs compared to that of OBDD, the counting argument is more sophisticated and more restrictive: it applies only to CNFs whose graphs are of constant degree. Due to this latter aspect, the target set of CNF instances requires a more delicate construction and reasoning.

Due to the space constraints, some proofs are either omitted or replaced by sketches. The complete proofs are provided in the appendix.

2 Preliminaries

In this paper by a *set of literals* we mean one that does not contain an occurrence of a variable and its negation. For a set S of literals we denote by $Var(S)$ the set of variables whose literals occur in S . If F is a Boolean function or its representation by a specified structure, we denote by $Var(F)$ the set of variables of F . A truth assignment to $Var(F)$ on which F is true is called a *satisfying assignment* of F . A set S of literals represents the truth assignment to $Var(S)$ where variables occurring positively in S (i.e. whose literals in S are positive) are assigned with *true* and the variables occurring negatively are assigned with *false*. We denote by F_S a function whose set of satisfying assignments consists of S' such that $S \cup S'$ is a satisfying assignment of F . We call F_S a *subfunction* of F .

We define a Non-deterministic Read Once Branching Program (NROBP) as a connected acyclic read-once *switching-and-rectifier network* [4]. That is, a NROBP Y implementing (realizing) a function F is a directed acyclic graph (with possible multiple edges) with one leaf, one root, and with some edges labelled by literals of the variables of F in a way that there is no directed path having two edges labelled with literals of the same variable. We denote by $A(P)$ the set of literals labeling edges of a directed path P of Y .

The connection between Y and F is defined as follows. Let P be a path from the root to the leaf of Y . Then any extension of $A(P)$ to the truth assignment of all the variables

of F is a satisfying assignment of F . Conversely, let A be a satisfying assignment of F . Then there is a path P from the root to the leaf of Y such that $A(P) \subseteq A$.

Remark. It is not hard to see that the traditional definition of NROBP as a deterministic ROBP with guessing nodes [5] can be thought as a special case of our definition (for any function that is not constant *false*): remove from the former all the nodes from which the *true* leaf is not reachable and relabel each edge with the appropriate literal of the variable labelling its tail (if the original label on the edge is 1 then the literal is positive, otherwise, if the original label is 0, the literal is negative).

We say that a NROBP Y is *uniform* if the following is true. Let a be a node of Y and let P_1 and P_2 be 2 paths from the root of Y to a . Then $Var(A(P_1)) = Var(A(P_2))$. That is, these paths are labelled by literals of the same set of variables. Also, if P is a path from the root to the leaf of Y then $Var(A(P)) = Var(F)$. Thus there is a one-to-one correspondence between the sets of literals labelling paths from the root to the leaf of Y and the satisfying assignments of F .

All the NROBPs considered in Sections 3-5 of this paper are uniform. This assumption does not affect our main result because an arbitrary NROBP can be transformed into a uniform one at the price of $O(n)$ times increase of the number of edges. For the sake of completeness, we provide the transformation and its correctness proof in the appendix. We use the construction described in the proof sketch of Proposition 2.1 of [6].

For our counting argument we need a special case of NROBP where *all* the edges are labelled, each node is of out-degree at most 2 and 2 out-edges of a node of degree exactly 2 are labelled with opposite literals of the same variable. We call this representation *normalized free binary decision diagram* (NFBDD).

We need additional terminology regarding NFBDD. We say that each non-leaf node a is *labelled* by the variable whose literals label its out-edges and denote this variable by $Var(a)$. Further on, we refer to the out-going edges of a labeled by, respectively, positive and negative literals of $Var(a)$ as *positive* and *negative* out-going edges of a . The heads of these edges are respective *positive* and *negative* out-neighbours of a (if both edges have the same head then these out-neighbours coincide). Note that given a labelling on nodes, there will be no loss of information if all the positive edges are labelled with 1 and all the negative edges are labelled with 0: the information about the labelling variable can be read from the tail of each edge and hence only the information about the sign of the labelling literal is needed. It follows that, for instance an OBDD with all the nodes from which the *yes*-leaf cannot be reached being removed is, in essence, an NFBDD. Consequently, any Boolean function that is not constant *false* can be represented by an NFBDD.

Figure 1 illustrates a NROBP and a NFBDD for a particular function.

Given a graph G , its *tree decomposition* is a pair (T, \mathbf{B}) where T is a tree and \mathbf{B} is a set of bags $B(t)$ corresponding to the vertices t of T . Each $B(t)$ is a subset of $V(G)$ and the bags obey the rules of *union* (that is, $\bigcup_{t \in V(T)} B(t) = V(G)$), *containment* (that is, for each $\{u, v\} \in E(G)$ there is $t \in V(T)$ such that $\{u, v\} \subseteq B(t)$), and *connectedness* (that is for each $u \in V(G)$, the set of all t such that $u \in B(t)$ induces a subtree of T). The *width* of (T, \mathbf{B}) is the size of the largest bag minus one. The treewidth of G is the smallest width of a tree decomposition of G .

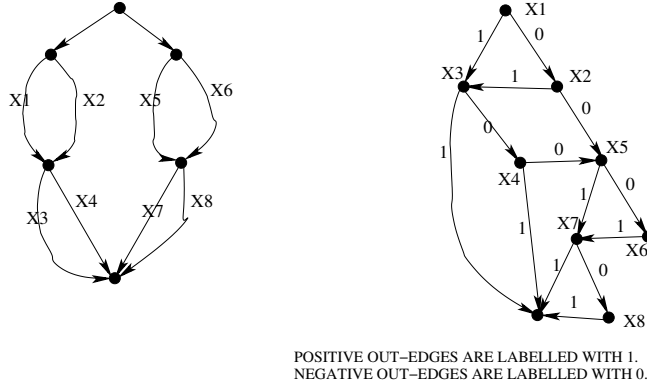


Fig. 1. NROBP and NFBDD for function $((x_1 \vee x_2) \wedge (x_3 \vee x_4)) \vee ((x_5 \vee x_6) \wedge (x_7 \vee x_8))$

Given a CNF ϕ , its *primal graph* has the set of vertices corresponding to the variables of ϕ . Two vertices are adjacent if and only if there is a clause of ϕ where the corresponding variables both occur.

3 The main result

A *monotone* 2-CNFs has clauses of the form $(x \vee y)$ where x and y are two distinct variables. Such CNFs can be put in one-to-one correspondence with graphs that do not have isolated vertices. In particular, let G be such a graph. Then G corresponds to a 2CNF $\phi(G)$ whose set of variables is $\{x_v | v \in V(G)\}$ and the set of clauses is $\{(x_u \vee x_v) | \{u, v\} \in E(G)\}$. It is not hard to see that G is a primal graph of $\phi(G)$, hence we can refer to the treewidth of G as the primal graph treewidth of $\phi(G)$. For $u \in V(G)$, denote by $Var(u)$ the variable of $\phi(G)$ corresponding to u and for $V' \subseteq V(G)$, let $Var(V') = \{Var(u) | u \in V'\}$. Conversely, let x be a literal of a variable of $\phi(G)$. Then the corresponding vertex of G is denoted by $Vert(x)$. If X' is a set of literals of variables of $\phi(G)$ then $Vert(X') = \{Vert(x) | x \in X'\}$.

The following theorem is the main result of this paper.

Theorem 1. *There is a constant c such that for each $k \geq 50$ there is an infinite class \mathbf{G} of graphs each of treewidth of at most k such that for each $G \in \mathbf{G}$, the smallest NROBP equivalent to $\phi(G)$ is of size at least $n^{k/c}$, where n is the number of variables of $\phi(G)$.*

In order to prove Theorem 1, we introduce the notion of *distant matching width* (DMW) of a graph and state two theorems proved in the subsequent two sections. One claims that if the max-degree of G is bounded then the size of a NROBP realizing $\phi(G)$ is exponential in the DMW of G . The other theorem claims that for each sufficiently large k there is an infinite class of graphs of a bounded degree and of treewidth at most k whose DMW is at least $b * \log n * k$ for some universal constant b . Theorem 1 will follow as an immediate corollary of these two theorems.

Definition 1. Matching width.

Let SV be a permutation of $V(G)$ of vertices of a graph. and let S_1 be a prefix of SV (i.e. all vertices of $SV \setminus S_1$ are ordered after S_1). The matching width of S_1 is the size of the largest matching consisting of the edges between S_1 and $V(G) \setminus S_1$ (we sometimes treat sequences as sets, the correct use will be always clear from the context). The matching width of SV is the largest matching width of a prefix of SV . The matching width of G , denoted by $mw(G)$, is the smallest matching width of a permutation of $V(G)$.

Remark. The above definition of matching width is a special case of a more general notion of *maximum matching width* as defined in [8]. In particular, our notion of matching width can be seen as a variant of maximum matching width of [8] where the tree T involved in the definition is a caterpillar. Also, [8] considers the notion of maximum induced matching width requiring that the ends of different edges of the witnessing matching are not adjacent. We need to impose a stronger constraint on the witnessing matching as specified below.

Definition 2. Distant matching

A matching M of G is distant if it is induced (no neighbours between vertices incident to distinct edges of M) and also no two vertices incident to distinct edges of M have a common neighbor.

Definition 3. Distant matching width

Distant matching width (DMW) is defined analogously to matching width with ‘matching’ replaced by ‘distant matching’. The DMW of graph G is denoted by $dmw(G)$. Put it differently, $dmw(G)$ equals the largest t such that for any permutation SV of $V(G)$ there is a partition V_1, V_2 into a prefix and a suffix such that there is a distant matching of size t consisting of edges with one end in V_1 and the other end in V_2 .

To illustrate the above notions recall that C_n and K_n respectively denote a cycle and a complete graph of n vertices. Then, for a sufficiently large n , $mw(C_n) = dmw(C_n) = 2$. On the other hand $mw(K_n) = \lfloor n/2 \rfloor$ while $dmw(K_n) = 1$.

Theorem 2. For each integer i there is a constant a_i such that for any graph G the size of NROBP realizing $\phi(G)$ is at least $2^{dmw(G)/a_x}$ where x is the max-degree of G .

Theorem 3. There is a constant b such that for each $k \geq 50$ there is an infinite class \mathbf{G} of graphs of degree at most 5 such that the treewidth of all the graphs of \mathbf{G} is at most k and for each $G \in \mathbf{G}$ the matching width is at least $(\log n * k)/b$ where $n = |V(G)|$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let \mathbf{G} be the class whose existence is claimed by Theorem 1. By theorem 2, for each $G \in \mathbf{G}$ the size of a NROBP realizing $\phi(G)$ is of size at least $2^{dmw(G)/a_5}$. Further on, by Theorem 3, $dmw(G) \geq (\log n * k)/b$, for some constant b . Substituting the inequality for $dmw(G)$ into $2^{dmw(G)/a_5}$, we get that the size of a NROBP is at least $2^{\log n * k/c}$ where $c = a_5 * b$. Replacing $2^{\log n}$ by n gives us the desired lower bound. ■

From now on, the proof is split into two independent parts: Section 4 proves Theorem 2 and Section 5 proves Theorem 3.

4 Proof of Theorem 2

Let S be an assignment to a subset of variables of $\phi(G)$ and $V' \subseteq V(G)$. We say that V' *covers* S if all the variables of $\text{Var}(V')$ occur positively in S . Furthermore, we call $V' \subseteq V(G)$ a *distant independent set* DIS of G if V' is an independent set of G and, in addition, no two vertices of V' have a common neighbour.

In order to prove Theorem 2, we first introduce Lemma 1 (proved in Section 4.1) stating that at least $2^{t/a}$ DISes are needed to cover all the satisfying assignments of $\phi(G)$ where a is a constant depending on the max-degree of G . After that we show that a NROBP Z of $\phi(G)$ always has a root-leaf (node) cut K such that each node u of the cut can be associated with a DIS of size $\text{dmw}(G)$ such that for all the root-leaf paths P passing through u , $A(P)$ (recall that $A(P)$ is the set of labels on the edges of P) is covered by this DIS. Since each satisfying assignment of $\phi(G)$ is $A(P)$ of some root-leaf path P and since P passes through a node of K (due to K being a root-leaf cut of Z), we conclude that all the satisfying assignments of $\phi(G)$ are covered by the considered family of DISes. Using Lemma 1, we will conclude that this set of DISes is large and hence the set K and, consequently, Z are large as well.

Lemma 1. *For each i there is a constant a_i such that for any t the number of DISes of G of size t needed to cover all the satisfying assignments of $\phi(G)$ is at least $2^{t/a_x}$ where x is the max-degree of G . Put it differently, if \mathbf{M} is a family of DISes of size t such that each satisfying assignment of $\phi(G)$ is covered by at least one element of \mathbf{M} then $|\mathbf{M}| \geq 2^{t/a_x}$*

Lemma 2. *Let P be a path from the root to the leaf of a NROBP realizing $\phi(G)$. Then P has a node $u(P)$ for which the following holds. Let P_1 be the prefix of P ending at $u(P)$ and let P_2 be the suffix of P starting at $u(P)$. Denote $\text{Vert}(A(P_1))$ and $\text{Vert}(A(P_2))$ by V_1 and V_2 , respectively. Then G has a distant matching M of size $\text{dmw}(G)$ such that one end of each edge of M is in V_1 and the other end is in V_2 .*

Proof. Let SL be the sequence of $\text{Var}(A(P))$ listed by the chronological order of the occurrence of respective literals on the edges of P being explored from the root to the leaf. Due to the uniformity and the read-onceness of Z , SL is just a permutation of the variables of $\phi(G)$. By definition of $\phi(G)$, SL corresponds to a permutation SV of $V(G)$. Moreover, for a prefix V_1 of SV , there is a partition of P into a prefix P_1 and a suffix P_2 such that $V_1 = \text{Vert}(A(P_1))$ and $V \setminus V_1 = \text{Vert}(A(P_2))$. Indeed, take a prefix P_1 including precisely the first $|V_1|$ -th labels by letting the final node of P_1 to be the head of the edge carrying the $|V_1|$ -th label. If the desired equalities are not satisfied then the vertices of SV are listed in an order different from the order of occurrence of the corresponding variables in SL , a contradiction. It remains to recall that by definition of $\text{dmw}(G)$, a witnessing partition V_1, V_2 exists for any permutation SV of $V(G)$ and to take the prefix and a suffix of P corresponding to such V_1 and V_2 . ■

The cut we will consider for the purpose of proving Theorem 2 will be the set of nodes $u(P)$ for all the paths P of Z from the root to the leaf. The next lemma will allow us to transform the matching associated with each vertex of this cut into a DIS by taking one vertex of each edge of this matching.

Lemma 3. *Let Z be a NROBP realizing $\phi(G)$ of a graph G . Let P be a path from the root to the leaf of Z and let a be a vertex of this path. Let P_1 be the prefix of P ending at a and let P_2 be the suffix of P beginning at a . Denote $\text{Vert}(A(P_1))$ and $\text{Vert}(A(P_2))$ by V_1 and V_2 respectively. Let $\{v_1, v_2\}$ be an edge of G such that $v_1 \in V_1$ and $v_2 \in V_2$. Then either $\{v_1\}$ covers all the assignments $A(P')$ such that P' is a root-leaf path of Z passing through a or this is true regarding $\{v_2\}$.*

Proof. Let x_1, x_2 be the respective variables of $\phi(G)$ corresponding to v_1 and v_2 . Recall that by definition, $\phi(G)$ contains a clause $(x_1 \vee x_2)$. Suppose that the statement of the lemma is not true. That is, there are 2 paths P' and P'' from the root to the leaf of Z , both passing through a and such that P' is not covered by v_1 and P'' is not covered by v_2 . Let P'_1, P'_2 be the prefix and suffix of P' with a being the final vertex of P'_1 and the initial vertex of P'_2 . Let P''_1 and P''_2 be the analogous partition of P'' .

Observe that due to the uniformity of Z , $\text{Vert}(A(P_1)) = \text{Vert}(A(P'_1))$. In particular, $v_1 \in \text{Vert}(A(P'_1))$ and hence the occurrence of x_1 in $A(P')$, in fact belongs to $A(P'_1)$. Since $\{v_1\}$ does not cover P' , $\neg x_1 \in A(P')$ and hence $\neg x_1 \in A(P'_1)$. Analogously, $\text{Ver}(A(P_1)) = \text{Ver}(A(P'_1))$ and hence $\text{Vert}(A(P'_1))$ does not contain v_2 leading to the conclusion that $\neg x_2 \in A(P'_2)$. By construction, $P'_1 \cup P'_2$ is a path from the root to the leaf of Z and hence $A(P'_1 \cup P'_2) = A(P'_1) \cup A(P'_2)$ is a satisfying assignment of $\phi(G)$. However, this is a contradiction since $A(P'_1) \cup A(P'_2)$ contains $\{\neg x_1, \neg x_2\}$ falsifying a clause of $\phi(G)$. ■

Now we are ready to prove Theorem 2.

Proof of Theorem 2. For each path P from the root to the leaf of Z , pick a vertex $u(P)$ as specified in Lemma 2. Let $\{u_1, \dots, u_q\}$ be the set of all such $u(P)$. By construction each of them is neither the root nor the leaf and each path from the root to the leaf passes through some u_i . So, they indeed constitute a root-leaf cut of Z . Further on, for each u_i specify a witnessing path P^i such that $u_i = u(P^i)$ and such that P^i_1 is the prefix of P^i ending at u_i and P^i_2 is the suffix of P^i beginning at u_i . By definition of $u(P_i)$, there is distant matching M_i of size $dmw(G)$ such that one end of each edge of M_i belongs to $\text{Vert}(A(P^i_1))$ and the other end belongs to $\text{Vert}(A(P^i_2))$. By Lemma 3 we can choose one end of each edge of M_i that covers $A(P')$ for all P' passing through u_i . Let B_i be the set of the chosen ends. By definition of a distant matching these vertices are mutually non-adjacent and do not have common neighbours. It follows that each B_i is a DIS of G of size $dmw(G)$. Moreover, by construction, each B_i covers $A(P')$ for all P' passing through u_i . It follows that each satisfying assignment A' of $\phi(G)$ is covered by some B_i . Indeed, by definition of NROBP, Z has a path P' from the root to the leaf such that $A(P') = A'$. Since $\{u_1, \dots, u_q\}$ is a root-leaf cut of Z , P' passes through some u_i . Consequently, $A' = A(P')$ is covered by B_i . It follows from Lemma 1 that $q \geq 2^{dmw(G)/a_x}$ where x is the max-degree of G , confirming the theorem. ■

4.1 Proof of Lemma 1

In order to prove Lemma 1, we assume that $\phi(G)$ is represented as a NFBDD Y . For each edge e of Y we assign weight $w(e)$ as follows. For a vertex a of Y with 2 leaving edges, the weight of each edge is 0.5. If a has only one leaving edge, the weight of

this edge is 1. The weight $w(P)$ of a path P of Y is defined as follows. If P consists of a single vertex then $w(P) = 1$. Otherwise $w(P)$ is the product of weights of its edges. Let \mathbf{P} be a set of paths. Then $w(\mathbf{P}) = \sum_{P \in \mathbf{P}} w(P)$ defines the weight of \mathbf{P} . The following proposition immediately follows from the non-negativity of weights.

Proposition 1. *Let $\mathbf{P}'_1, \dots, \mathbf{P}'_x$ be a sets of paths of Y . Then $w(\bigcup_{i=1}^x \mathbf{P}'_i) \leq \sum_{i=1}^x w(\mathbf{P}'_i)$*

Let a be a node of Y and let \mathbf{P}_a be the set of all paths from a to the leaf of Y . Then the following can be easily noticed.

Proposition 2. $w(\mathbf{P}_a) = 1$.

Let $S \subseteq V(G)$. Let \mathbf{P}_a^S be the subset of \mathbf{P}_a consisting of all P such that $A(P)$ is covered by S . We will show that if S is a DIS of G and G is of bounded degree then $w(\mathbf{P}_{rt}^S)$ is exponentially small in $|S|$ where rt is the root of Y . Then we will note that if S_1, \dots, S_q are DISes such that each satisfying assignment is covered by one of them then $\mathbf{P}_{rt}^{S_1} \cup \dots \cup \mathbf{P}_{rt}^{S_q} = \mathbf{P}_{rt}$ and hence $w(\mathbf{P}_{rt}^{S_1}) + \dots + w(\mathbf{P}_{rt}^{S_q}) \geq w(\mathbf{P}_{rt}^{S_1} \cup \dots \cup \mathbf{P}_{rt}^{S_q}) = 1$. Consequently, q must be exponentially large in $|S|$, implying the lemma. This weighted counting approach is inspired by a probabilistic argument as in e.g. [6].

We denote by $Vert_a$ the set of vertices of G corresponding to the variables that have not been assigned by a path from the root to a .

We denote by $Free_a$ the subset of $Vert_a$ consisting of all vertices v such that there is a path $P \in \mathbf{P}_a$ with $\neg Var(v) \in A(P)$. (This is only possible if no label $\neg Var(u)$ occurs on a path from the root to a such that u is a neighbour of v . That is, v is ‘free’ in the sense that it is not constrained by such an occurrence.)

For $v \in V(G)$ we denote by $ld_a(v)$ the number of neighbours of v in $Vert_a$ (‘ld’ stands for ‘local degree’).

For $B \subseteq V(G)$, we define $rw_a(B)$ as follows (‘rw’ stands for ‘relative weight’). If $B = \emptyset$ then $rw_a(B) = 1$. Otherwise, let $v \in B$. Then $rw_a(B) = (1 - 2^{-(ld_a(v)+1)}) * rw_a(B \setminus \{v\})$. For a non-empty B , $rw_a(B)$ can be seen as $\prod_{v \in B} (1 - 2^{-(ld_a(v)+1)})$.

The following is our main technical argument.

Lemma 4. *Let a be a node of Y and let $B \subseteq Free_a$ be a DIS of G . Then $w(\mathbf{P}_a^B) \leq rw_a(B)$.*

In the rest of the section we prove Lemma 1 and then provide a proof of Lemma 4.

Proof of Lemma 1. Denote the root of Y by rt . It is not hard to see that $Free_{rt} = V(G)$ (for each variable of $\phi(G)$ there is a satisfying assignment where this variable appears negatively), hence Lemma 4 applies to \mathbf{P}_{rt}^B for an arbitrary DIS B of G . Moreover, $ld_{rt}(v)$ is simply $d(v)$, the degree of v in G . Therefore, it follows from Lemma 4 that $w(\mathbf{P}_{rt}^B) \leq \prod_{v \in B} (1 - 2^{-(d(v)+1)})$. Since $d(v) \leq x$ (recall that x denotes the max-degree of G), $\prod_{v \in B} (1 - 2^{-(d(v)+1)}) \leq \prod_{v \in B} (1 - 2^{-(x+1)}) = (1 - 2^{-(x+1)})^t$ where $t = |B|$. Let B_1, \dots, B_q be DISes of size t that cover all the satisfying assignments of $\phi(G)$. It follows that $\mathbf{P}_{rt}^{B_1} \cup \dots \cup \mathbf{P}_{rt}^{B_q} = \mathbf{P}_{rt}$. Indeed, the left-hand side is contained in the right-hand side by definition, so let $P \in \mathbf{P}_{rt}$. Then, by definition, of Y , $A(P)$ is a satisfying assignment of $\phi(G)$. By definition of B_1, \dots, B_q , there is some

B_i covering $A(P)$. Then it follows that $P \in \mathbf{P}_{rt}^{B_i}$. Combining propositions 1 and 2, we obtain: $1 = w(\mathbf{P}_{rt}) = w(\bigcup_{i=1}^q \mathbf{P}_{rt}^{B_i}) \leq \sum_{i=1}^q w(\mathbf{P}_{rt}^{B_i}) \leq q * (1 - 2^{-(x+1)})^t$

It follows that $q \geq (\frac{1}{1-2^{-(x+1)}})^t$. Clearly, for each x there is a constant a_x such that $\frac{1}{1-2^{-(x+1)}}$ can be represented as $2^{1/a_x}$. Hence the bound $q \geq 2^{t/a_x}$ follows. ■

To prove Lemma 4, we need a number of auxiliary statements provided below.

Lemma 5. *Let a be a non-leaf node of Y having only one out-neighbour. Then this out-neighbour is positive.*

Lemma 6. *Let a be a node of Y and let a' be an out-neighbour of a . Denote $\text{Vert}(\text{Var}(a))$ by v and let $B \subseteq \text{Free}_a$. Then the following statements hold.*

- If $v \in B$ then $B \setminus \{v\} \subseteq \text{Free}_{a'}$.
- If there is $w \in B$ such that $\{v, w\} \in E(G)$ and a' is a negative out-neighbour of a then $B \setminus \{w\} \subseteq \text{Free}_{a'}$.
- In all other cases, $B \subseteq \text{Free}_{a'}$.

Let (a, a') be an edge of Y and let P be a path of Y starting at a' . Then $(a, a') + P$ denotes the path obtained by concatenating (a, a') and P . Let \mathbf{P} be a set of paths all starting at a' . Then $(a, a') + \mathbf{P} = \{(a, a') + P \mid P \in \mathbf{P}\}$.

Proposition 3. $w((a, a') + \mathbf{P}) = w(a, a') * w(\mathbf{P})$.

Lemma 7. *Let a be a node of Y . Denote $\text{Vert}(\text{Var}(a))$ by v . Let B be a DIS of G . Then the following statements are true.*

- Assume that $v \in B$. Then $\mathbf{P}_a^B \subseteq (a, a') + \mathbf{P}_{a'}^{B \setminus \{v\}}$ where a' is the positive out-neighbour of a .
- Otherwise, $\mathbf{P}_a^B \subseteq \bigcup_{a' \in N_Y^+(a)} ((a, a') + \mathbf{P}_{a'}^B)$, where $N_Y^+(a)$ is the set of out-neighbours of a .

Lemma 8. *Let a be a node of Y , let a' be an out-neighbour of a , and let B be a DIS of G . Denote $\text{Vert}(\text{Var}(a))$ by v . Then the following statements hold.*

- Assume that $v \in B$. Then $rw_{a'}(B \setminus \{v\}) = rw_a(B) / (1 - 2^{-(ld_a(v)+1)})$.
- Assume there is $w \in B$ such that $\{v, w\} \in E(G)$. Then $rw_{a'}(B \setminus \{w\}) = rw_a(B) / (1 - 2^{-(ld_a(w)+1)})$ and $rw_{a'}(B) = rw_a(B) * \frac{1-2^{-ld_a(w)}}{(1-2^{-(ld_a(w)+1)})}$.
- If none of the above assumptions is true then $rw_{a'}(B) = rw_a(B)$.

Proof of Lemma 4. The proof is by induction on the reverse topological ordering of the nodes of Y (leaves first and if a non-leaf node is considered, the lemma is assumed correct for all its out-neighbours). Let a be a leaf of Y . Clearly, $\text{Free}_a = \emptyset$ and hence we can only consider the set \mathbf{P}_a^\emptyset consisting of a single path including node a itself. It follows that $w(P_a^\emptyset) = 1$. On the other hand, $rw_a(\emptyset) = 1$ by definition. Hence the lemma holds in the considered case.

Assume now that a is not a leaf and denote $\text{Vert}(\text{Var}(a))$ by v .

Suppose first that $v \in B$. Since $v \in \text{Free}_a$, there is a path $P^* \in \mathbf{P}_a$ such that $\text{Var}(v)$ occurs negatively in P^* . That is P^* contains a node a^* such that $\text{Var}(a^*) =$

$Var(v)$ and the leaving edge of a^* included in P^* is the negative one. Due to the read-onceness, the only node of P^* whose associated variable is $Var(v)$ is a . Consequently, a has a leaving negative edge. It follows from Lemma 5 that a has 2 out-neighbours and hence the weight of each leaving edge is 0.5.

Let a' be the positive out-neighbour of a . Combining Lemma 7 and Proposition 3, we obtain, $w(\mathbf{P}_a^B) \leq w((a, a') + \mathbf{P}_{a'}^{B \setminus \{v\}}) = w(a, a') * w(\mathbf{P}_{a'}^{B \setminus \{v\}}) = 0.5 * w(\mathbf{P}_{a'}^{B \setminus \{v\}})$.

By Lemma 6, $B \setminus \{v\} \subseteq Free_{a'}$. By the induction assumption and Lemma 8, $w(\mathbf{P}_{a'}^{B \setminus \{v\}}) \leq rw_{a'}(B \setminus \{v\}) = rw_a(B) / (1 - 2^{-(ld_a(v)+1)})$. It follows that $w(\mathbf{P}_a^B) \leq 0.5 * rw_a(B) / (1 - 2^{-(ld_a(v)+1)})$. Since $1 - 2^{-(ld_a(v)+1)} \geq 0.5$, $w(\mathbf{P}_a^B) \leq rw_a(B)$.

Suppose that v is a neighbour of some $w \in B$. Assume first that a has only one out-neighbour a' . According to Lemma 5, a' is a positive out-neighbour. Combining Lemma 7, Proposition 3, and taking into account that $w(a, a') = 1$, we obtain the following. $w(\mathbf{P}_a^B) \leq w((a, a') + \mathbf{P}_{a'}^B) = w(\mathbf{P}_{a'}^B)$. By Lemma 6, $B \subseteq Free_{a'}$. By the induction assumption combined with Lemma 8, we obtain: $w(\mathbf{P}_a^B) \leq w(\mathbf{P}_{a'}^B) \leq rw_{a'}(B) = rw_a(B) * \frac{1 - 2^{-(ld_a(w))}}{(1 - 2^{-(ld_a(w)+1)})}$. The numerator in the last item is smaller than the denominator and hence $w(\mathbf{P}_a^B) \leq rw_a(B)$ follows.

Assume now that in addition to a' , a has the negative out-neighbour a'' . According to Lemma 7, $\mathbf{P}_a^B \subseteq ((a, a') + \mathbf{P}_{a'}^B) \cup ((a, a'') + \mathbf{P}_{a''}^B)$. Since any assignment covered by B is also covered by a subset of B , $\mathbf{P}_{a''}^B \subseteq \mathbf{P}_{a''}^{B \setminus \{w\}}$ and hence $\mathbf{P}_a^B \subseteq ((a, a') + \mathbf{P}_{a'}^B) \cup ((a, a'') + \mathbf{P}_{a''}^{B \setminus \{w\}})$. Note that $B \subseteq Free_{a'}$ and $B \setminus \{w\} \subseteq Free_{a''}$ by Lemma 6. Combining the induction assumption with Proposition 1, with Lemma 8, and with the fact that $w((a, a')) = w((a, a'')) = 0.5$, we obtain, $\mathbf{P}_a^B \leq 0.5 * rw_a(B) * \frac{1 - 2^{-(ld_a(w))}}{(1 - 2^{-(ld_a(w)+1)})} + 0.5 * rw_a(B) / (1 - 2^{-(ld_a(w)+1)}) = 0.5rw_a(B) \frac{2 - 2^{-(ld_a(w))}}{(1 - 2^{-(ld_a(w)+1)})} = 0.5 * 2 * rw_a(B) \frac{1 - 2^{-(ld_a(w)+1)}}{(1 - 2^{-(ld_a(w)+1)})} = rw_a(B)$.

Suppose that none of the previous assumptions occur. By Corollary 6, $B \subseteq Free_{a'}$ for any out-neighbour of a . By the induction assumption, combined with Lemma 8, $\mathbf{P}_{a'}^B \leq rw_{a'}(B)$. Hence, by Lemma 7 combined with Proposition 3 and Proposition 1, we obtain, $w(\mathbf{P}_a^B) \leq \sum_{a' \in N_Y^+(a)} w(a, a') * rw_a(B) = rw_a(B)$. ■

5 Proof of Theorem 3

In order to prove Theorem 3, we consider graphs $T(H)$ where T is a tree and H is an arbitrary graph. Then $T(H)$ is a graph having disjoint copies of H in one-to-one correspondence with the vertices of T . For each pair t_1, t_2 of adjacent vertices of T , the corresponding copies are connected by making adjacent the pairs of *same* vertices of these copies. Put it differently, we can consider H as a labelled graph where all vertices are associated with distinct labels. Then for each edge $\{t_1, t_2\}$ of T , edges are introduced between the vertices of the corresponding copies having the same label. An example of this construction is shown on Figure 2.

Denote by T_r a complete binary tree of height (root-leaf distance) r . The following structural lemma is the critical component of the proof of Theorem 3.

Lemma 9. *Let p be an arbitrary integer and let H be an arbitrary connected graph of $2p$ vertices. Then for any $r \geq \lceil \log p \rceil$, $mw(T_r(H)) \geq (r + 1 - \lceil \log p \rceil)p/2$*

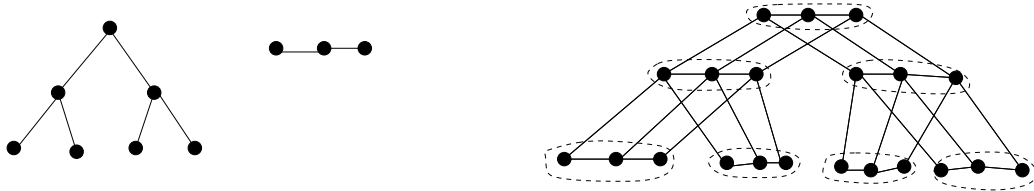


Fig. 2. Graphs from the left to the right: T_3 , P_3 , $T_3(P_3)$. The dotted ovals surround the copies of P_3 in $T_3(P_3)$.

Before proving Lemma 9, let us show how Theorem 3 follows from it.

Sketch proof of Theorem 3. First of all, let us identify the class \mathbf{G} . Recall that P_x a path of x vertices. Let $0 \leq y \leq 3$ be such that $k - y + 1$ is divided by 4. The considered class \mathbf{G} consists of all $G = T_r(P_{\frac{k-y+1}{2}})$ for $r \geq 5\lceil \log k \rceil$. It can be observed that the max-degree of the graphs of \mathbf{G} is 5 and their treewidth is at most k .

Taking into account that starting from a sufficiently large r compared to k , $r = \Omega(\log(n/k))$ can be seen as $r = \Omega(\log n)$, the lower bound of Lemma 9 can be stated as $mw(G) = \Omega(\log n * k)$. Finally, we observe that for bounded-degree graphs $mw(G)$ and $dmw(G)$ are linearly related and conclude that a lower bound on $mw(G)$ implies the analogous lower bound on $dmw(G)$. ■

The following lemma is an auxiliary statement for Lemma 9.

Lemma 10. *Let T be a tree consisting of at least p vertices. Let H be a connected graph of at least $2p$ vertices. Let V_1, V_2 be a partition of $V(T(H))$ such that both partition classes contain at least p^2 vertices. Then $T(H)$ has a matching of size p with the ends of each edge belong to distinct partition classes.*

Proof of Lemma 9. The proof is by induction on r . The first considered value of r is $\lceil \log p \rceil$. After that r will increment in 2. In particular, for all values of r of the form $\lceil \log p \rceil + 2x$, we will prove that $mw(T_r(H)) \geq (x+1)p$ and, moreover, for each permutation SV of $V(T_r(H))$, the required matching can be witnessed by a partition of SV into a suffix and a prefix of size at least p^2 each. Let us verify that the lower bound $mw(T_r(H)) \geq (x+1)p$ implies the lemma. Suppose that $r = \lceil \log p \rceil + 2x$ for some non-negative integer x . Then $mw(G) \geq (x+1)p = ((r - \lceil \log p \rceil)/2 + 1)p > (r - \lceil \log p \rceil + 1)p/2$. Suppose $r = \lceil \log p \rceil + 2x + 1$. Then $mw(G) = mw(T_r(H)) \geq mw(T_{r-1}(H)) \geq (x+1)p = ((r - \lceil \log p \rceil - 1)/2 + 1)p = (r - \lceil \log p \rceil + 1)p/2$.

Assume that $r = \lceil \log p \rceil$ and let us show the lower bound of p on the matching width. T_r contains at least $2^{\lceil \log p \rceil + 1} - 1 \geq 2^{\log p + 1} - 1 = 2p - 1 \geq p$ vertices. By construction, H contains at least $2p$ vertices. Consequently, for each ordering of vertices of T_r we can specify a prefix and a suffix of size at least p^2 (just choose a prefix of size p^2). Let V_1 be the set of vertices that got to the prefix and let V_2 be the set of vertices that got to the suffix. By Lemma 10 there is a matching of size at least p consisting of edges between V_1 and V_2 confirming the lemma for the considered case.

Let us now prove the lemma for $r = \lceil \log p \rceil + 2x$ for $x \geq 1$. Specify the center of T_r as the root and let T^1, \dots, T^4 be the subtrees of T_r rooted by the grandchildren of the root. Clearly, all of T^1, \dots, T^4 are copies of T_{r-2} . Let SV be a se-

quence of vertices of $V(T_r(H))$. Let SV^1, \dots, SV^4 be the respective sequences of $V(T^1(H)), \dots, V(T^4(H))$ ‘induced’ by SV (that is their order is as in SV). By the induction assumption, for each of them we can specify a partition SV_1^i, SV_2^i into a prefix and a suffix of size at least p^2 each witnessing the conditions of the lemma for $r-2$. Let u_1, \dots, u_4 be the last respective vertices of SV_1^1, \dots, SV_1^4 . Assume w.l.o.g. that these vertices occur in SV in the order they are listed. Let SV', SV'' be a partition of SV into a prefix and a suffix such that the last vertex of SV' is u_2 . By the induction assumption we know that the edges between $SV_1^2 \subseteq SV'$ and $SV_2^2 \subseteq SV''$ form a matching M of size at least xp . In the rest of the proof, we are going to show that the edges between SV' and SV'' whose ends do not belong to any of SV_1^2, SV_2^2 can be used to form a matching M' of size p . The edges of M and M' do not have joint ends, hence this will imply existence of a matching of size $xp + p = (x+1)p$, as required.

The sets $SV' \setminus SV_1^2$ and $SV'' \setminus SV_2^2$ partition $V(T_r(H)) \setminus (SV_1^2 \cup SV_2^2) = V(T_r(H)) \setminus V(T^2(H)) = V([T_r \setminus T^2](H))$. Clearly, $T_r \setminus T^2$ is a tree. Furthermore, it contains at least p vertices. Indeed, T^2 (isomorphic to T_{r-2}) has p vertices just because we are at the induction step and T_r contains at least 4 times more vertices than T^2 . So, in fact, $T_r \setminus T^2$ contains at least $3p$ vertices. Furthermore, since u_1 precedes u_2 , the whole SV_1^1 is in SV' . By definition, SV_1^1 is disjoint with SV_1^2 and hence it is a subset of $SV' \setminus SV_1^2$. Furthermore, by definition, $|SV_1^1| \geq p^2$ and hence $|SV' \setminus SV_1^2| \geq p^2$ as well. Symmetrically, since $u_3 \in SV''$, we conclude that $SV_2^3 \subseteq SV'' \setminus SV_2^2$ and due to this $|SV'' \setminus SV_2^2| \geq p^2$.

Thus $SV' \setminus SV_1^2$ and $SV'' \setminus SV_2^2$ partition $V([T_r \setminus T^2](H))$ into classes of size at least p^2 each and the size of $T_r \setminus T^2$ is at least $3p$. Thus, according to Lemma 10, there is a matching M' of size at least p created by edges between $SV' \setminus SV_1^2$ and $SV'' \setminus SV_2^2$, confirming the lemma, as specified above ■

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A Transformation of a NROBP into a uniform one

Let Z be the NROBP being transformed and let F be the function of n variables realized by Z . Let a_1, \dots, a_m be the non-leaf nodes of Z being ordered topologically. We show that there is a sequence $Z_{a_1} = Z, Z_{a_2}, \dots, Z_{a_m}$ such that each Z_{a_i} for $i > 1$ is a NROBP of F obtained from $Z_{a_{i-1}}$ by subdividing the in-coming edges of a_i by adding at most n nodes and $O(n)$ edges to each such an in-coming edge. Moreover, the edges of any two paths P_1 and P_2 from the root of Z_{a_i} to a_i or to any node topologically preceding a_i are labelled with literals of the same set of variables. Observe that since each edge has only one head, say a_j , it is subdivided only once, namely during the construction of Z_{a_j} . Hence the number of new added edges of Z_{a_m} is $O(n)$ per edge of Z and hence the size of Z_{a_m} is $O(n)$ times larger than the size of Z .

Regarding Z_{a_1} this existence statement is vacuously true so assume $i > 1$. Denote by $AllVar(a_i)$ the set of all variables whose literals label edges of paths of $Z_{a_{i-1}}$ from the root to a_i .

For each in-neighbour a' of a_i , we transform the edge (a', a_i) as follows. Let P be a path from the root of $Z_{a_{i-1}}$ to a_i passing through (a', a_i) . Let x^1, \dots, x^q be the elements of $AllVar(a_i) \setminus Var(A(P))$. We subdivide (a', a_i) as follows. We introduce new nodes a'_1, \dots, a'_q and let $a_{q+1} = a_i$. Then instead (a', a_i) we introduce an edge (a', a'_1) carrying the same label as (a', a_i) (or no label in case (a', a_i) carries no label). Then, for each $1 \leq i \leq q$ we introduce two edges (a'_i, a'_{i+1}) carrying labels x^i and $\neg x^i$, respectively.

Let us show that the edges of any two paths P_1 and P_2 from the root of Z_{a_i} to a_i are labeled with literals of the same set of variables. Let a' be an in-neighbour of a_i in $Z_{a_{i-1}}$. By the induction assumption, any two paths from the root to a' are labelled with literals of the same set of variables. It follows that as a result any two paths from the root to a_i passing through a' are labelled by literals of the same set of variables, namely $AllVar(a_i)$. Since this is correct for an arbitrary choice of a' , we conclude that in Z_{a_i} any two paths from the root to a_i are labelled with $AllVar(a_i)$, that is with literals of the same set of variables. Observe that the paths to the nodes of Z preceding a_i are not affected so the ‘uniformity’ of paths regarding them holds by the induction assumption. Regarding the new added nodes on the subdivided edge (a', a_i) the uniformity clearly follows from the uniformity of paths from the root to a' .

To verify read-onceness of Z_{a_i} , let P' be a path from the root to the leaf of Z_{a_i} . Taking into account the induction assumption, the only reason why P' may contain two edges labelled by literals of the same variable is that P' is obtained from a path P of $Z_{a_{i-1}}$ by subdivision of an edge (a', a_i) of this path. By construction the variables of the new labels put on (a', a_i) do not occur on the prefix of P ending at a_i . Furthermore, by definition of $AllVar(a_i)$ the variable x each new label, in fact occurs in some path of $Z_{a_{i-1}}$ from the root to a_i and hence, by the read-onceness, x does not occur on any path starting from a_i . It follows that the variables of the new labels do not occur on the suffix of P' starting at a_i . Taking into account that all the new labels of (a', a_i) are literals of distinct variables, the read-onceness of P' , and hence the read-onceness of Z_{a_i} , due to the arbitrary choice of P' , follow. Thus we know now that Z_{a_i} is a NROBP.

It remains to verify that Z_{a_i} indeed realizes F . Let P' be a path of Z_{a_i} from the root to the leaf. Then $A(P')$ is an extension of $A(P)$ of some path P of Z_{a_i} . By the

induction assumption, any extension of $A(P)$ is a satisfying assignment of F , hence so is $A(P')$. Conversely, for each satisfying assignment A of F we can find a path P of $Z_{a_{i-1}}$ such that $A(P) \subseteq A$. If an edge of path P is subdivided then the new labels are opposite literals on multiple edges. So, for every such multiple edge we can choose one edge carrying the literal occurring in A and obtain a path P' such that $A(P') \subseteq A$.

For the leaf node we do a similar transformation but this time add new labels on the in-coming edges of the leaf so that the set of labels on each path from the root to the leaf is a set of literals of $Var(F)$. A similar argumentation to the above shows that the resulting structure is indeed a uniform NROBP realizing F . Clearly the size of the resulting NROBP remains $O(n)$ times larger than the size of Z .

B Proofs of auxiliary statements for Lemma 4

Proof of Lemma 5. Assume the opposite that let P be a path from the root to the leaf of Y passing through a . It follows that in $A(P)$, $Var(a)$ occurs negatively. Due to the monotonicity of $\phi(G)$, replacing $\neg Var(a)$ by $Var(a)$ in $A(P)$ produces another satisfying assignment A' of $\phi(G)$. Let P^a be the prefix of P ending at a . Since $A(P^a) \subseteq A'$, by definition of a uniform NROBP, there is a path P' from a to the leaf of Y such that $A(P^a \cup P') = A'$. Since $Var(a)$ occurs positively in A' this is only possible if the successor of a in P' is its positive out-neighbour in contradiction to our assumption of its non-existence. ■

To prove Lemma 6, we need an auxiliary statement.

Lemma 11. *Let Y be a NFBDD realizing $\phi(G)$ and let a be a node of Y . Let P_1 be a path from the root to a . Denote $Vert(A(P_1))$ by Vrt . Let $A' \subseteq A(P_1)$ be the set of all negative literals of $A(P_1)$ and denote $Vert(A')$ by $Vng(P_1)$. Then $Free_a = V(G) \setminus (Vrt \cup N_G(Vng(P_1)))$.*

Proof. Let $v \in Free_a$. Then Y has a path P_2 from a to the leaf such that $Var(v)$ occurs negatively in $A(P_2)$. Due to read-onceness of Y , $Var(v)$ does not occur in $A(P_1)$, hence $v \notin Vrt$. Assume that v is a neighbour of some $u \in Vng(P_1)$. By definition of Y , $A(P_1 \cup P_2)$ is a satisfying assignment of $\phi(G)$ containing $\{\neg Var(u), \neg Var(v)\}$ which is a contradiction since $\phi(G)$ contains a clause $(Var(u) \vee Var(v))$. Thus $v \notin N_G(Vng(P_1))$ and thus we have verified that $Free_a \subseteq V(G) \setminus (Vrt \cup N_G(Vng(P_1)))$.

Conversely, let $v \in V(G) \setminus (Vrt \cup N_G(Vng(P_1)))$. It follows that $Var(v)$ does not occur in $A(P_1)$ and that $Var(v)$ does not occur in the same clause of $\phi(G)$ with any of $Var(Vng(P_1))$. Consequently, there is a satisfying assignment A' of $\phi(G)$ such that $A(P_1) \subseteq A'$ and $Var(v)$ occurs negatively in A' : just assign positively the rest of the variables. By definition of a uniform NROBP, there is path P_2 from a to the leaf of Y such that $A(P_1 \cup P_2) = A'$. Clearly $A(P_1 \cup P_2) = A(P_1) \cup A(P_2)$ and $Var(v)$ occurs negatively in $A(P_2)$. Hence $v \in Free_a$ and thus we have confirmed that $V(G) \setminus (Vrt \cup N_G(Vng(P_1))) \subseteq Free_a$, completing the lemma. ■

Proof of Lemma 6. It is not hard to see that in each case the considered subset of B is a subset of $Vert_{a'}$. By Lemma 11, it remains to set a path P' from the root of Y to a' and to verify that in each item the considered subset of B does not have neighbours in $Vng(P')$ (as defined in Lemma 11). Let P be a path from the root to a and let P'

be a path obtained by appending (a, a') to the end of P . Clearly $Vng(P')$ is $Vng(P)$ plus, possibly, $Vert(Var(a)) = v$ in case a' is a negative out-neighbour of a . Since $B \subseteq Free_a$, it follows from Lemma 11 that B is not adjacent with $Vng(P)$. Hence, it remains to verify that in each case the considered subset of B is not adjacent with v . This is certainly true in the first case because B is an independent set and hence $B \setminus \{v\}$ is not adjacent with v . In the second case due to being B a DIS, v does not have neighbours in B other than w and hence v is not adjacent with $B \setminus \{w\}$. In the third case either a' is a positive out-neighbour of a and hence $v \notin Vng(P')$ or v is not adjacent to B (otherwise we obtain the second case). In any case, B is not adjacent with $Vng(P')$.

Proof of Proposition 3. Indeed, $w((a, a') + \mathbf{P}) = \sum_{P \in \mathbf{P}} w((a, a') + P) = \sum_{P \in \mathbf{P}} (w(a, a') * w(P)) = w(a, a') * \sum_{P \in \mathbf{P}} w(P) = w(a, a') * w(\mathbf{P})$, as required. ■

Proof of Lemma 7. Suppose that $v \in B$. Let $P \in \mathbf{P}_a^B$. Clearly the element a' following a is an out-neighbour of a . However, if a' is the negative out-neighbour of a then $Var(v)$ occurs negatively in $A(P)$ and hence B does not cover P , a contradiction. It remains to assume that a' is the positive out-neighbour of a . Hence, \mathbf{P}_a^B can be represented as $(a, a') + \mathbf{P}'$ where \mathbf{P}' is a set of paths starting at a' . It remains to show that $\mathbf{P}' \subseteq \mathbf{P}_{a'}^{B \setminus \{v\}}$. Let $P' \in \mathbf{P}'$. Then $A(P) = A((a, a')) \cup A(P')$ is covered by B (here we admit a notational abuse identifying an edge with a path). However, the only variable occurring positively in $A((a, a'))$ is $Var(v)$. It remains to assume that $Var(B \setminus \{v\})$ occur positively in P' , that is P' is covered by $B \setminus \{v\}$. Thus we have proved the first statement.

Suppose $v \notin B$. Clearly, \mathbf{P}_a^B is the union of all $(a, a') + \mathbf{P}'$ where a' is an out-neighbour of a and \mathbf{P}' is some set of paths starting at a' . Let $P' \in \mathbf{P}'$. Then $A((a, a') + P')$ is covered by B , however $A((a, a'))$ is not covered by any subset of B . It remains to assume that P' is covered by B and hence $\mathbf{P}' \subseteq \mathbf{P}_{a'}^B$. ■

Proof of Lemma 8. For the first item, notice that $Vert_{a'} = Vert_a \setminus \{v\}$ and that, due to being B an independent set, no vertex of $B \setminus \{v\}$ is adjacent to v . It follows that that the neighbours of each $u \in B \setminus \{v\}$ in $Vert_{a'}$ are exactly the same as in $Vert_a$ and hence $ld_a(u) = ld_{a'}(u)$. It follows that the factor contributed by each vertex of $B \setminus \{v\}$ to $rw_{a'}(B \setminus \{v\})$ and to $rw_a(B \setminus \{v\})$ is the same. That is, $rw_{a'}(B \setminus \{v\}) = rw_a(B \setminus \{v\}) = rw_a(B)/(1 - 2^{-(ld_a(v)+1)})$, as required.

For the second item, notice that, due to B being a DIS, v is not a neighbour of any vertex of B other than w . It follows that that the neighbours of each $u \in B \setminus \{w\}$ in $Vert_{a'} = Vert_a \setminus \{v\}$ are exactly the same as in $Vert_a$ and hence $ld_a(u) = ld_{a'}(u)$. It follows that the factor contributed by each vertex of $B \setminus \{w\}$ to $rw_{a'}(B \setminus \{w\})$ and to $rw_a(B \setminus \{w\})$ is the same. That is, $rw_{a'}(B \setminus \{w\}) = rw_a(B \setminus \{w\}) = rw_a(B)/(1 - 2^{-(ld_a(w)+1)})$, as required. On the other hand, w has one neighbour less in $Vert_{a'}$ than in $Vert(a)$. That is, $ld_{a'}(w) = ld_a(w) - 1$. Clearly, $rw_{a'}(B)$ can be obtained by multiplying $rw_{a'}(B \setminus \{w\})$ by the factor contributed by w . That is $rw_{a'}(B) = rw_{a'}(B \setminus \{w\}) * (1 - 2^{-ld_a(w)}) = rw_a(B) * \frac{1 - 2^{-ld_a(w)}}{(1 - 2^{-(ld_a(w)+1)})}$.

For the last item it is easy to see that the local degrees of vertices of B are the same regarding a' and a and hence they contribute the same factor and the desired equality follows. ■

C Proofs of statements for Theorem 3

The next lemma is an auxiliary statement needed for proving Lemma 10.

Lemma 12. *Suppose the vertices of $T(H)$ are partitioned into 2 subsets V_1 and V_2 . Let L be a subset of vertices of H such that $|L| = t$. Suppose there are two copies H_1 and H_2 of H such that for each $u \in L$ the copies of vertex u in H_1 and H_2 belong to different partition classes. Then $T(H)$ has matching of size t with the ends of each edge lying in different partition classes*

Proof. Let v_1 and v_2 be the respective vertices of T corresponding to H_1 and H_2 . Let p be the path between v_1 and v_2 in T . Then for each $u \in L$ there are two consecutive vertices v'_1 and v'_2 of this path with respective copies H'_1 and H'_2 such that the copy u'_1 of u in H'_1 belongs to the same partition class as the copy u_1 of u in H_1 and the copy u'_2 of u in H'_2 belongs to the same partition class as the copy u_2 of u in H_2 . By construction, $T(H)$ has an edge $\{u'_1, u'_2\}$ which we choose to correspond to u . Let $L = \{u^1, \dots, u^t\}$ and consider the set of edges as above corresponding to each u^i . By construction, both ends of the edge corresponding to each u^i are copies of u^i and also these ends correspond to distinct partition classes. It follows that these edges do not have joint ends and indeed constitute a desired matching of size t ■

Proof of Lemma 10. The proof is under assumption that T contains *exactly* p vertices. Indeed, otherwise, such a tree can be obtained by an iterative removal of the copies of H associated with vertices having degree 1. Clearly, any matching of the resulting restricted graph will also be a matching of the original graph and the lower bound on the sizes of the partition classes will be preserved as well.

Assume first that each copy of H corresponding to a vertex of T contains vertices of both partition classes. Since H is a connected graph, for each copy we can specify an edge with one end in V_1 and the other end in V_2 . These edges belong to disjoint copies of H , hence none of these edges have a common end. Since there are p copies of H , we have the desired matching of size p .

Assume now that there is a vertex u of T such that the copy H_1 of H corresponding to u contains vertices of only one partition class. Assume w.l.o.g. that this class is V_1 . Then there is a vertex v of T such that the copy H_2 of H corresponding to v contains at least p vertices of V_2 . Indeed, otherwise, at most $p - 1$ vertices per p copies will not make p^2 vertices altogether. Let L be the set of vertices of H whose copies in H_2 belong to V_2 . By assumption, all the copies of L in H_1 belong to V_1 . By Lemma 12, H_1 and H_2 witness the existence of matching of size p with ends of each edge belonging to distinct partition classes. ■

In order to prove Theorem 3, we need an auxiliary proposition.

Proposition 4. *For a graph G with maxdegree c , $dmw(G) \geq mw(G)/(2c^2 + 2c + 1)$.*

Proof. For each ordering of vertices of G take the partition witnessing $mw(G)$ and let M be a witnessing matching of size $mw(G)$. Let $\{u_i, v_i\}$ be an edge of M . It is not hard to see that the number of vertices v whose open neighbourhood intersects with that of $\{u_i, v_i\}$ is at most $(2c + 2)c$. Indeed, $|N[u_i, v_i]| \leq 2c + 2$. If for some vertex v , $N[v] \cap N[u_j, v_j] \neq \emptyset$ then $v \in N[N[u_j, v_j]]$. Clearly, $|N[N[u_j, v_j]]| \leq (2c + 2)c$ as required.

Now, let us create a distant matching M^* out of M . Take $\{u_1, v_1\}$ to M^* and remove it from M together with at most $2c^2 + 2c$ pairs whose open neighborhood may intersect with $N[u_1, v_1]$. Until M is not empty take the survived $\{u_i, v_i\}$ of the smallest index i and perform the same operation. It clearly follows by construction that M^* is a distant matching. Let us compute its size. On each step the number of pairs removed from M is at most $2c^2 + 2c + 1$, so the number of iterations of adding pairs to M^* and hence the number of such pairs is at least $mw(G)/(2c^2 + 2c + 1)$.

We conclude that for each permutation of vertices of G there is a partition witnessing the desired distant matching, as required. ■

Proof of Theorem 3. First of all, let us identify the class \mathbf{G} . Recall that P_x a path of x vertices. Further on, let $0 \leq y \leq 3$ be such that $k - y + 1$ is divided by 4. The considered class \mathbf{G} consists of all $G = T_r(P_{\frac{k-y+1}{2}})$ for $r \geq 5\lceil \log k \rceil$.

Let us show that the treewidth of the graphs of \mathbf{G} is bounded by k . Consider the following tree decomposition of $G = T_r(H = P_{\frac{k-y+1}{2}})$. The tree is T_r . Consider T_r as the rooted tree with the centre being the root. The bag of each vertex includes the vertices of the copy of H associated with this vertex plus the copy of the parent (for a non-root vertex). The properties of tree decomposition can be verified by a direct inspection. The size of each bag is at most $k - y + 1$, hence the treewidth is at most $k - y \leq k$.

Observe that max-degree of the graphs of \mathbf{G} is 5. Indeed, consider a vertex v of $G \in \mathbf{G}$ that belongs to a copy of H associated with a vertex x of some T_r . Inside its copy of H , v is adjacent to at most 2 vertices. Outside its copy of H , v is adjacent to vertices in the copies of H associated with the neighbours of x , precisely one neighbour per copy. Vertex x is adjacent to at most 3 vertices of T_r . It follows that v has at most 3 neighbours outside its copy of H .

It follows from Proposition 4 that DMW and the matching width of graphs of \mathbf{G} are linearly related. Therefore, it is sufficient to obtain the desired lower bound on the matching width. This is done in the next paragraph.

Let us reformulate the lower bound of $mw(G)$ in terms of $\log n$ and k where $n = V(G)$. Notice that p used in Lemma 9 can be expressed as $(k - y + 1)/4$. Hence, the lower bound on the matching width can be seen as $(r - \lceil \log(\frac{k-y+1}{4} + 1) \rceil) * (k - y + 1)/8$. This lower bound can be immediately simplified by noticing that by the choice of k and y , $(k - y + 1)/8 \geq k/16$ and $\lceil \log(\frac{k-y+1}{4}) \rceil \leq \lceil \log k \rceil$. Hence, $(r - \lceil \log k \rceil + 1)k/16$ can serve as a lower bound on $mw(G)$. To draw the connection between n and r , notice that $n = (2^{r+1} - 1)(k - y + 1)/2$. It follows that $r + 1 = \log(\frac{n}{(k-y+1)/2} + 1)$. In particular, it follows that $r + 1 \geq \log n - \log k \geq \log n - \lceil \log k \rceil$. It follows that $r + 1$ in the lower bound can be replaced by $\log n - \lceil \log k \rceil$ and the new lower bound is $(\log n - 2\lceil \log k \rceil)k/16$. Consequently, for $\log n \geq 5\lceil \log k \rceil$ the lower bound can be represented as $(\log n * k)/32$ which is the form needed for the theorem. It remains to observe that $r \geq 5\lceil \log k \rceil$ implies $\log n \geq 5\lceil \log k \rceil$. By the above reasoning, $r \geq 5\lceil \log k \rceil$ implies $\log(\frac{n}{(k-y+1)/2} + 1) \geq 5\lceil \log k \rceil$. By our choice of $k \geq 50$, $\log(n/20 + 1) \geq \log(\frac{n}{(k-y+1)/2} + 1) \geq 5\lceil \log k \rceil$. By construction of G and the choice of r , $n \geq 2^{r+1} - 1 \geq k^5 - 1 \geq k$, the last inequality follows from the choice of k , hence $n \geq 50$. In particular, it follows that $n \geq n/20 + 1$. Hence $\log n \geq \log(n/20 + 1) \geq 5\lceil \log k \rceil$. ■